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Abstract

We derive a generic expression for the generating function (GF) of the particle-number probability distribution (PNPD) for a simple reaction diffusion model that belongs to the directed percolation universality class. Starting with a single particle on a lattice, we show that the GF of the PNPD can be written as an infinite series of cumulants taken at zero momentum. This series can be summed up into a complete form at the level of a mean-field approximation. Using the renormalization group techniques, we determine logarithmic corrections for the GF at the upper critical dimension. We also find the critical scaling form for the PNPD and check its universality numerically in one dimension. The critical scaling function is found to be universal up to two non-universal metric factors.

PACS numbers: 05.10.Gg, 04.40.−a

1. Introduction

In recent decades, the statistical mechanics of non-equilibrium models characterized by large-scale spacetime fluctuations and absorbing phase transition has been studied extensively with numerical and analytical tools [1]. The directed percolation (DP) universality class covers a large variety of models of this type, which have short-range interactions, a single absorbing state and a scalar order parameter [2, 3]. The DP conjecture has been extended to models with multiple components and even multiple absorbing states, unless there is an additional symmetry and a conservation law. Nevertheless, we should mention that there is at least one exception in the case of more elaborated interactions [4–6].

Besides various numerical techniques, the response functional formalism combined with the renormalization group (RG) techniques is one of the powerful analytical tool to explore the asymptotic properties near criticality for these models [7–10]. The DP universality is described

by the Reggeon field theory which has been investigated in details by many researchers [1]. Its upper critical dimension is known to be 4 and the values of the critical exponents were obtained up to $\mathcal{O}(\epsilon^2)$ with space dimension $d = 4 - \epsilon$ [2, 12]. The leading logarithmic corrections at the upper critical dimension are also available in the literature [13, 14] and tested numerically [15]. Recently, the same field-theoretical formalism was successfully applied to identify and calculate the survival probability and the mean number of particles, which are the central quantities of interest in the damage-spreading (DS) type dynamics [14, 16].

The aim of this paper is to explore in the same framework a more general and useful quantity in the DS-type dynamics, the generating function (GF) of the particle-number probability distribution (PNPD). In principle, all moments of the number of particles including the survival probability can be derived from the GF. We start from a lattice description of a reaction diffusion model that belongs to the DP universality class. Then, we use the bosonic operator formalism [8] to represent the master equation and the associated field theory to define the generating function (GF) of the PNPD. With a single particle initially, we derive the GF of the PNPD as an infinite series of multi-point Green functions. This series can be summed up explicitly under the tree (mean-field) approximations into a complete form.
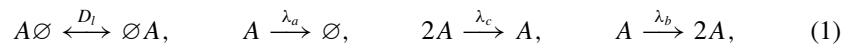
Assuming the generic critical scaling property of the Green functions, we obtain the critical scaling form of the PNPD. Using the RG techniques, we calculate the leading logarithmic corrections for the critical GF at the upper critical dimension. Our results are consistent with recent reports for the survival probability and the average number of particles by Janssen and Stenull [14, 16].

The conventional RG analysis suggests that the critical scaling function for the PNPD should be universal up to two non-universal metric factors whose microscopic details of the models are integrated into. We check this universality numerically in one dimension for two variants of bosonic DP models and also for the hard-core DP models where the site occupancy of particles is restricted to one. Numerical simulations show that all the models form a universality very nicely at criticality with only two adjusting parameters.

The paper is organized as follows: section 2 introduces the model and defines the GF for the PNPD. Section 3 is devoted to the derivation of the series expansion of the GF in terms of connected Green functions. In section 4, we derive the mean-field solution and present the logarithmic corrections to the scaling in the critical state at the upper critical dimension for the first three moments of the PNPD. In section 5, we analyse the scaling form of the critical PNPD, using the RG equations and test it numerically in one dimension for both bosonic models and hard-core models. We present the conclusion in section 6 and the technical details for the calculation of the logarithmic corrections in the appendix.

2. General formalism

We consider a reaction diffusion model (the Gribov process) [2, 3] that belongs to the DP universality. It is defined on a d-dimensional lattice with N^d sites which may be either vacant or occupied by particles. The evolution rule is summarized in the stoichiometric notation as

$$A\varnothing \xleftrightarrow{D_l} \varnothing A, \qquad A \xrightarrow{\lambda_a} \varnothing, \qquad 2A \xrightarrow{\lambda_c} A, \qquad A \xrightarrow{\lambda_b} 2A, \qquad (1)$$

where D_l is the nearest neighbour hopping (diffusion) rate, λ_a is the annihilation rate for a particle, λ_c is the coagulation rate for two particles at a site and λ_b is the branching rate.

A microscopic state of the system is defined by the set of particle numbers at each lattice site: $\mathbf{n} = \{n_i\}$. The time evolution of the probability distribution $P(\mathbf{n}, t)$ can be described by

the master equation for the state vector $|\Psi(t)\rangle = \sum_{\mathbf{n}} P(\mathbf{n}, t)|\mathbf{n}\rangle$ at time t:

$$\frac{\mathrm{d}|\Psi\rangle}{\mathrm{d}t} = -H|\Psi\rangle. \tag{2}$$

The evolution operator H can be written in terms of bosonic operators as

$$H = D_l \sum_{\langle ij \rangle} (a_i^\dagger - a_j^\dagger)(a_i - a_j) - \lambda_a \sum_i (a_i - a_i^\dagger a_i)$$
$$- \lambda_c \sum_i (a_i^\dagger a_i^2 - a_i^{\dagger 2} a_i^2) - \lambda_b \sum_i (a_i^{\dagger 2} a_i - a_i^\dagger a_i), \tag{3}$$

where $a_i (a_i^\dagger)$ is the annihilation (creation) operator of bosonic particles, satisfying the commutation relation $[a_i, a_j^\dagger] = \delta_{ij}$.

The average of a quantity O at time t is computed by the formula

$$\langle O \rangle_t = \langle 0| \exp\left(\sum_i a_i\right) \hat{O} \, \mathrm{e}^{-tH} |\Psi(0)\rangle \tag{4}$$

for a given initial state $|\Psi(0)\rangle$ [7–9], where $|0\rangle$ represents the vacuum state and $\hat{O}$ is the operator representation of the quantity O. Similarly, one can easily derive the probability that the system has n particles on the lattice at time t [17] as

$$p(n, t) = \frac{1}{n!} \langle 0| \left(\sum_i a_i\right)^n \mathrm{e}^{-tH} |\Psi(0)\rangle, \tag{5}$$

and its generating function (GF) is defined as

$$F(s, t) = \sum_{n=0} s^n p(n, t) = \langle 0| \exp\left(\sum_i s a_i\right) \mathrm{e}^{-tH} |\Psi(0)\rangle. \tag{6}$$

3. The expansion of the generating function

It is well known that the calculation of the average becomes simpler by moving the factor $\exp(\sum_i a_i)$ in equation (4) to the rightmost position. Since $\mathrm{e}^a a^\dagger = (a^\dagger + 1)\mathrm{e}^a$, this move results in shifting $a^\dagger \to a^\dagger + 1$ in the operators $\hat{O}$ and H. We use the same trick to calculate the GF in equation (6), and then

$$F(s, t) = \langle 0| \exp\left(\sum_i (s - 1)a_i\right) \mathrm{e}^{-tH_s} \exp\left(\sum_i a_i\right) |\Psi(0)\rangle, \tag{7}$$

where the shifted Hamiltonian $H_s(a^\dagger, a) = H(a^\dagger + 1, a)$.

Through the standard coherent-state path-integral formulation [8, 9], one can find the shifted action $S(\bar{\phi}, \phi)$ such that the path integral is weighed by e^{-S} as

$$S(\bar{\phi}, \phi) = \int \mathrm{d}t \left[\sum_i \bar{\phi}_i \partial_t \phi_i + H_s(\bar{\phi}, \phi) \right] = \int \mathrm{d}t \left[\sum_i \bar{\phi}_i \partial_t \phi_i + D_l \sum_{\langle ij \rangle} (\bar{\phi}_i - \bar{\phi}_j)(\phi_i - \phi_j) \right.$$
$$\left. + \sum_i \left\{ (\lambda_a - \lambda_b)\bar{\phi}_i \phi_i + \lambda_c \bar{\phi}_i \phi_i^2 - \lambda_b \bar{\phi}_i^2 \phi_i + \lambda_c \bar{\phi}_i^2 \phi_i^2 \right\} \right]. \tag{8}$$

For the calculation of the asymptotic properties, the action is taken formally in the continuum limit with the transformations

$$\sum_i \rightarrow b^{-d} \int \mathrm{d}x^d, \qquad f_{x_i+be_j} - f_{x_i} \rightarrow b\nabla_{x_j} f(x),$$

$$\phi_i(t) \rightarrow b^d \phi(x,t), \qquad \bar{\phi}_i(t) \rightarrow \bar{\phi}(x,t), \tag{9}$$

$$D = b^2 D_l, \qquad \mu = \lambda_a - \lambda_b, \qquad g' = b^d \lambda_c, \qquad g'' = \lambda_b,$$

where b is the lattice constant and e_j is the unit vector along the j axis. In the continuum limit, the action reads

$$S = \int \mathrm{d}x \, \mathrm{d}t \left[\bar{\phi}(x,t) \partial_t \phi(x,t) + D(\nabla \bar{\phi}(x,t))(\nabla \phi(x,t)) \right.$$

$$\left. + \mu \bar{\phi}(x,t) \phi(x,t) + \frac{1}{2} g' \bar{\phi}(x,t) \phi^2(x,t) - \frac{1}{2} g'' \bar{\phi}^2(x,t) \phi(x,t) \right], \tag{10}$$

where the term $g'\bar{\phi}^2\phi^2$ is dropped as being irrelevant. The coupling constants g' and g'' can be reduced to one coupling constant $g = \sqrt{g'g''}$ by rescaling the fields as follows: $\phi \rightarrow \alpha\phi, \bar{\phi} \rightarrow \alpha^{-1}\bar{\phi}$, with $\alpha = \sqrt{g''/g'}$. The rescaled 'symmetric' action is then invariant under the time-reversal transformation: $t \rightarrow -t, \phi \rightarrow -\bar{\phi}, \bar{\phi} \rightarrow -\phi$.

Consider the GF with one particle initially: $|\Psi(0)\rangle = a_0^\dagger |0\rangle$. Then, we have

$$F(s,t) = \langle 0| \exp\left(\sum_i (s-1)a_i\right) \mathrm{e}^{-tH_s}\left(a_0^\dagger + 1\right)|0\rangle$$

$$= 1 + \langle 0| \exp\left(\sum_i (s-1)a_i\right) \mathrm{e}^{-tH_s} a_0^\dagger |0\rangle$$

$$= 1 + \sum_{p=1} \frac{(s-1)^p}{p!} \sum_{i_1,\ldots,i_p} \langle 0|a_{i_1}(t)\cdots a_{i_p}(t) \, \mathrm{e}^{-tH_s} a_0^\dagger(0)|0\rangle, \tag{11}$$

where we used the relations $H_s|0\rangle = 0$ and $\langle 0|H_s = 0$ because each term in H_s has $a^\dagger$ to the left and a to the right. In the path-integral formulation, the above correlation functions are simply connected Green functions (cumulants) with the shifted action in equation (8). In the continuum limit with the symmetric action, the GF is written as

$$F(s,t) = 1 + \sum_{p=1} \frac{(s-1)^p}{p!} \alpha^{p-1} \int \mathrm{d}x_1, \ldots, \mathrm{d}x_p G^{(p,1)}(x_1,t,\ldots,x_p,t;0,0), \tag{12}$$

where

$$G^{(p,1)}(x_1,t,\ldots,x_p,t;0,0) = \langle \phi(x_1,t)\cdots\phi(x_p,t)\bar{\phi}(0,0)\rangle \tag{13}$$

are the connected Green functions for the symmetric action and the factor $\alpha = \sqrt{g''/g'}$ stems from the rescaling of the fields.

Close to the critical point, we can use the renormalized Green functions of which the scaling behaviour is known. Then, we have

$$G^{(p,1)}(x_1,t,\ldots,x_p,t;0,0) = Z_\phi^{p/2} Z_{\bar{\phi}}^{1/2} G_R^{(p,1)}(x_1,t,\ldots,x_p,t;0,0)$$

$$= Z_\phi^{p/2} Z_{\bar{\phi}}^{1/2} t^{-p\frac{\beta}{\nu_\parallel} - \frac{\beta'}{\nu_\parallel}} \mathcal{G}^{(p,1)}(x_1/t^{1/z},\ldots,x_p/t^{1/z}), \tag{14}$$

where $Z_\phi, Z_{\bar{\phi}}$ are the fields renormalization factors that are equal for the time-reversal symmetric theory and we have used the scaling property of the renormalized Green functions close to the critical point as

$$G_R^{(p,1)}(x_1,t_1,\ldots,x_p,t_p;x_0,t_0) = |\mu|^{p\beta+\beta'} H(|\mu|^{\nu_\perp} x_1, |\mu|^{\nu_\parallel} t_1, \ldots), \tag{15}$$

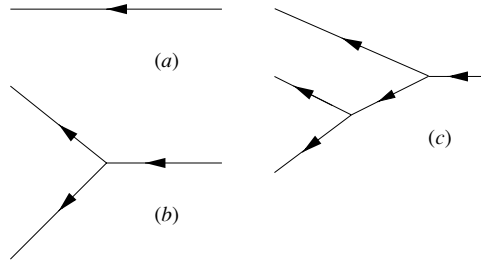**Figure 1.** (*a–c*) The first three tree diagrams used in the mean-field approximation.

where μ is the deviation from the critical point, and β, β', $\nu_\parallel$, $\nu_\perp$ and z are the standard critical exponents defined in [1].

The generating function can be written now as

$$
\begin{aligned}
\bar{F}(s, t) &\equiv F(s, t) - 1 \\
&= \sum_{p=1} \frac{(s-1)^p}{p!} \alpha^{p-1} Z_\phi^{p/2} Z_{\bar{\phi}}^{1/2} t^{-p(\frac{\beta}{\nu_\parallel} - \frac{d}{z}) - \frac{\beta'}{\nu_\parallel}} A^{(p,1)} \\
&= t^{-\frac{\beta'}{\nu_\parallel}} Z_\phi^{1/2} Z_{\bar{\phi}}^{1/2} \left[\sum_{p=1} \frac{(s-1)^p}{p!} \left(\alpha Z_\phi^{1/2} \right)^{p-1} t^{p(\frac{d}{z} - \frac{\beta}{\nu_\parallel})} A^{(p,1)} \right],
\end{aligned}
\tag{16}
$$

where

$$
A^{(p,1)} = \int dx_1, \ldots, dx_p \mathcal{G}^{(p,1)}(x_1, \ldots, x_p).
\tag{17}
$$

This expression shows that $\bar{F}(s, t)$ can be renormalized by multiplication with $Z_\phi^{-1/2} Z_{\bar{\phi}}^{-1/2}$ after the parameter α is renormalized as $\alpha_R = Z_\phi^{1/2} \alpha$.

4. Mean-field approximation and logarithmic corrections

4.1. Mean-field solution

In equation (16), the GF is expressed as a formal series whose convergence property depends on the coefficients $A^{(p,1)}$. As a first step to study this series, we calculate the mean-field (MF) solution for the GF. In this approximation, we consider only the tree diagrams for the multi-point correlation function $G^{(p,1)}$ in equation (12).

The lowest order approximation of the $(p, 1)$ Green function can be obtained by connecting $p - 1$ vertices of type $\bar{\phi}^2 \phi$ with free two-point Green functions, as shown in figure 1 where each line is associated with the free two-point $(1, 1)$ Green function in the momentum space as

$$
G(k, t) = \theta(t) \exp(-t(Dk^2 + \mu)).
\tag{18}
$$

For the sum of tree diagrams we can easily obtain an integral recurrence equation for the GF by resuming the diagrams after the first vertex from the right in the tree, which is

$$
\bar{F}(s, t) = \bar{F}(s, 0) e^{-\mu t} + \frac{1}{2} g'' \int_0^t dt' \bar{F}(s, t - t')^2 G(k = 0, t'),
\tag{19}
$$

that is equivalent with the differential equation

$$
\frac{d}{dt} \bar{F}(s, t) = -\mu \bar{F}(s, t) + \frac{1}{2} g'' \bar{F}(s, t)^2.
\tag{20}
$$

With the single particle initial condition $F(s, 0) = s$ ($\bar{F}(s, 0) = s - 1$), one can find the exact solution for the GF as

$$F(s, t) = 1 + \frac{(s - 1) e^{-\mu t}}{1 - (s - 1)\frac{g''}{2\mu}(1 - e^{-\mu t})}. \tag{21}$$

By inverting the GF in equation (6), we find the PNPD as

$$p(n, t) = \frac{e^{-\mu t}}{\left(1 + \frac{g''}{2\mu}(1 - e^{-\mu t})\right)^2} \left(\frac{\frac{g''}{2\mu}(1 - e^{-\mu t})}{1 + \frac{g''}{2\mu}(1 - e^{-\mu t})}\right)^{n-1}, \tag{22}$$

for $n \geqslant 1$. At criticality ($\mu = 0$), the PNPD behaves in the asymptotic limit (large t) as

$$p(n, t) \approx \left(\frac{2}{g'' t}\right)^2 \exp\left(-\frac{2(n - 1)}{g'' t}\right). \tag{23}$$

The survival probability $p_s(t)$ that the system is active at time t can be derived from the expression $p_s(t) = 1 - p(0, t) = 1 - F(0, t)$. From equation (21), we find

$$p_s(t) = \frac{e^{-\mu t}}{1 + \frac{g''}{2\mu}(1 - e^{-\mu t})}. \tag{24}$$

We also find the mean number of particles $N(t) = \langle n \rangle = \sum_n n p(n, t) = \partial F/\partial s|_{s=1}$ as

$$N(t) = e^{-\mu t}. \tag{25}$$

Similarly, all higher moments $\langle n^k \rangle$ can be obtained by differentiating the GF with respect to s and taking the $s \to 1$ limit.

At criticality, the survival probability decays algebraically as

$$p_s(t) = \frac{1}{1 + g'' t/2} \approx \frac{2}{g''} t^{-1}, \tag{26}$$

while $N(t)$ remains a constant of unity. This is consistent with other MF predictions for the dynamic scaling exponents: $\delta_{\text{MF}} = 1$ and $\eta_{\text{MF}} = 0$ [1].

Near criticality, we find, for the large t limit, that $p_s(t) \approx (2\mu/g'') e^{-\mu t}$ in the absorbing side ($\mu \gtrsim 0$) and $p_s(t) \approx (2|\mu|/g'')(1 + e^{-|\mu| t})$ in the active side ($\mu \lesssim 0$). These results are fully consistent with the recent results by Janssen [16].

4.2. Logarithmic corrections to critical scaling

At the critical dimension, where the coupling constant is dimensionless, the MF solution (21) can be improved systematically at large time with the help of the renormalization group equations by adding the one-loop correction. A detailed account of this procedure has been presented recently in [14]. In our case we use one-loop approximation for the GF, combined with the two-loop Wilson functions taken from literature. The details of the calculation for one-loop GF and its asymptotic scaling derived with the help of RG equations are presented in the appendix. In this section we present, for the critical state, in a parametric form, the leading order expressions of the survival probability, $p_s(t)$, and of the first two moments of the particle-number distribution, $N(t)$, $N_2(t)$, which might be of interest for numerical experiments.

From equation (A.23) we have at large t

$$\begin{aligned} p_s(t) &= -\bar{F}(0, t) \\ &= \frac{4N'}{\bar{D}t} w^{-1} \exp(c_0 w^2 + O(w^4))(1 + (3 \ln X_0 - 9/2)w^2) + O(w^{-5/6} t^{-2}), \end{aligned} \tag{27}$$

$$N(t) = \frac{\partial \bar{F}(s,t)}{\partial s}\bigg|_{s=1}$$
$$= N'w^{-1/3}\exp(c_1 w^2 + O(w^4))(1 + (\ln X_0 + 1)w^2), \tag{28}$$

$$N_2(t) = \frac{\partial^2 \bar{F}(s,t)}{\partial s^2}\bigg|_{s=1} + N(t)$$
$$= \frac{1}{2}N'\bar{D}tw^{1/3}\exp(c_2 w^2 + O(w^4))(1 - (\ln X_0 - 4)w^2) + O(w^{-1/3}), \tag{29}$$

where w depends on time t through equations (A.25), (A.26). In the above expressions, the universal constants c_0, c_1, c_2 are defined in terms of the coefficients of the Wilson functions, equations (A.7)–(A.9), and have the values

$$c_0 = \frac{1}{48}\left(7 - 34\ln\frac{4}{3}\right) \tag{30}$$

$$c_1 = -\frac{1}{144}\left(25 + 322\ln\frac{4}{3}\right) \tag{31}$$

$$c_2 = -\frac{1}{144}\left(71 + 542\ln\frac{4}{3}\right). \tag{32}$$

The remaining N', $\bar{D}$, X_0 and t_0 are non-universal constants whose detailed origins can be explored in the appendix.

We make the observation that equations (27), (28) are equivalent with the expressions of [14] after the change of variable $w \to (w/8)^{1/2}$.

5. Critical scaling functions for the PNPD

Since the GF of the PNPD is renormalizable, see equation (16), we can use the RG equation to find its scaling properties in the critical state. At large t, we can extract the scaling property of the PNPD through the GF scaling behaviour.

A dimensional analysis with equation (16) suggests the following form for the GF:

$$\bar{F}(s,t) = yf(D\kappa^2 t, y\alpha\kappa^{-d/2}, u), \tag{33}$$

where κ^{-1} is an arbitrary length, u is the dimensionless coupling constant (see the appendix) and $y = s - 1$.

The independence of the bare $\bar{F}$ on κ yields the renormalization group equation

$$\kappa\frac{\partial}{\partial\kappa}(yZ_\phi f(D\kappa^2 t, y\alpha\kappa^{-d/2}, u)) = 0 \tag{34}$$

or, in more details, we have

$$\left(\gamma_\phi + \kappa\frac{\partial}{\partial\kappa} + \gamma_D D\frac{\partial}{\partial D} + \gamma_\alpha\alpha\frac{\partial}{\partial\alpha} + \beta(u)\frac{\partial}{\partial u}\right)yf(D\kappa^2 t, y\alpha\kappa^{-d/2}, u) = 0 \tag{35}$$

where $\gamma_\phi = \kappa\partial_\kappa\ln Z_\phi$, $\gamma_D = \kappa\partial_\kappa\ln Z_D$, $\gamma_\alpha = \gamma_\phi/2$ and $\beta(u) = \kappa\partial_\kappa u$, with Z_ϕ, Z_D the standard renormalization factors [8].

Equation (33) gives the following relations for the differential operators:

$$\kappa\frac{\partial}{\partial\kappa} = 2t\frac{\partial}{\partial t} - \frac{d}{2}\alpha\frac{\partial}{\partial\alpha}, \qquad D\frac{\partial}{\partial D} = t\frac{\partial}{\partial t}, \qquad \alpha\frac{\partial}{\partial\alpha} = y\frac{\partial}{\partial y} - 1, \tag{36}$$

hence the generic solution at the fixed point is

$$\bar{F}(s,t) = t^{-\frac{\gamma_\phi^*}{2+\gamma_D^*}} y \bar{f}\left(y t^{\frac{d/2-\gamma_\phi^*/2}{2+\gamma_D^*}}\right),$$ (37)

where starred parameters are values of corresponding γ functions at the fixed point.

Using the standard RG technique [18], the function $\bar{f}$ up to $O(\epsilon)$ and a multiplicative non-universal amplitude can be obtained by replacing in equation (A.6) $u^2 \rightarrow \epsilon/12$, $rt \rightarrow -C_y y t^{(d/2-\gamma_\phi^*/2)/(2+\gamma_D^*)}$ and $t/\bar{t} \rightarrow 1$; where $C_y > 0$ is a second non-universal amplitude, $\gamma_\phi^* = -\epsilon/6$ and $\gamma_D^* = -\epsilon/12$ up to $O(\epsilon)$ [10].

Since the difference between $p(n,t)$ and $p(n+1,t)$ for large t is small, the generating function $\bar{F}$ can be approximated by a Laplace transform with the parameter $\omega = -\ln s$. Applying the inverse Laplace transform in which we approximate $y \approx -\omega$, we obtain

$$p(n,t) = t^{-d/z} \bar{g}\left(\frac{n}{t^{d/(2z)+\eta/2}}\right)$$ (38)

where we have introduced the exponents $z = 2 + \gamma_D^*$ and $\eta = -\gamma_\phi^*/(2+\gamma_D^*)$ defined by scaling properties of the two-point Green function [8]. Equation (38) along with the relation $p_s(t) = \int dn\, p(n,t)$ gives the survival exponent $\delta = (d/z - \eta)/2$, which is the well-known hyperscaling relation.

The scaling analysis predicts that the critical PNPD at large t has the form

$$p(n,t) = \frac{C_2}{t^{\eta+2\delta}} g\left(C_1 \frac{n}{t^{\eta+\delta}}\right),$$ (39)

where C_1, C_2 are two non-universal metric factors which depend on the values of the microscopic parameters and g is the universal scaling function. The two-scale factor universality is worthwhile to be mentioned at this point. Since we are interested in the dynamic behaviour of the system, there are, in general, three independent metric factors which are related to the number of particles, the time and the distance from the critical point, respectively. The third metric factor does not appear in equation (39), because the system is at the critical point. This should be compared to the two-scale factor universality in equilibrium [19].

In one dimension, it is relatively easy to test numerically the above scaling form. We performed Monte Carlo simulations for the bosonic model [20] described by equation (1) at two points in the parameter space $(D_l, \lambda_a, \lambda_b, \lambda_c)$ on the critical manifold: $D_l = 0.1$ and 0.2, $\lambda_a = (1 - 2D_l)p_c$, $\lambda_b = (1 - 2D_l)(1 - p_c)$, $\lambda_c = 0.5$ with $p_c = 0.110\,56(1)$ and $0.133\,45(4)$, respectively. We measured the PNPD at two different times $t = 4000$ and 8000 for 2×10^7 samples. As shown in figure 2, we overlap the scaling functions almost perfectly by varying the constants C_1 and C_2, which implies that the scaling function g is universal. The numerical values of the exponents η and δ are taken from [21]: $\eta = 0.313\,686$ and $\delta = 0.159\,464$. In the same figure, we also plotted the scaling function of a variant model which has a pair annihilation process $2A \rightarrow \varnothing$ instead of the pair coagulation $2A \rightarrow A$. This one also overlaps perfectly with the others.

As pointed out in the introduction, there are many other models belonging to the DP universality class. In the field-theoretical formalism, they differ by extra irrelevant coupling terms in the action besides the terms in equation (10). Among these models, of a particular interest are the hard-core models where the dynamic rules accept only one or zero particle per site. As such examples, we have collected the PNPD data for the contact process [22] and the directed bond percolation [1] in one dimension at the same times as for the previous data. Their scaling functions can also be overlapped very well with those of the bosonic models in the same manner. We have noted a minute deviation between the two scaling functions
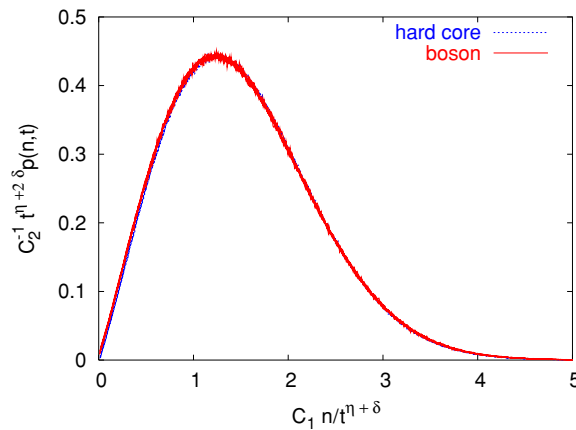
Figure 2. Critical scaling function for the PNPD in one dimension with full lines for boson models (pair annihilation and pair coagulation) and with dashed lines for hard-core particle models (the contact process and the bond-directed percolation). The curve corresponding to the contact process is chosen for reference, and the other data sets are plotted for the best overlap with the reference curve by varying the constants C_1 and C_2.

(This figure is in colour only in the electronic version)

at small values of the scaling variable $C_1 n / t^{\eta+\delta}$, but a more detailed analysis shows that this deviation is due to corrections to scaling which decay very slowly in this region.

6. Conclusions

We have shown that the generating function of the particle-number probability distribution of a DP model can be expressed as a series of cumulants of the associated field theory. Using the recurrence properties of the tree diagrams, we have explicitly calculated the mean-field solution and the logarithmic corrections for the PNPD at the upper critical dimension.

From the renormalization group analysis, we have shown that the scaling function for the PNPD is universal in the critical state and the microscopic details of the model are included in two non-universal metric factors. We have tested the universality numerically in one dimension for both boson and hard-core particle models. Numerical simulations show that all models form the universality with only two adjusting parameters in the critical state.

Acknowledgments

Appendix. One-loop calculation and logarithmic corrections

The logarithmic corrections to the critical scaling at the upper critical dimension can be obtained using the RG equation (35). For the calculation of logarithmic correction of the critical GF up to the sub-leading order, we need the expression of the critical GF up to one-loop order and the Wilson functions up to two-loop order. The Wilson functions have been
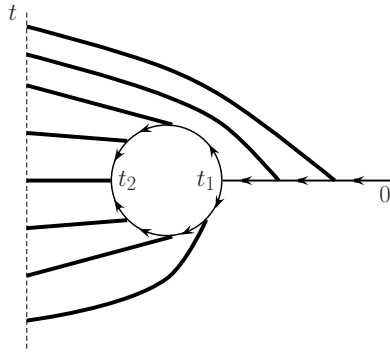
Figure 3. A generic one-loop diagram. The thick line represents the MF solution of the GF whose final time is t.

$$K(t_2, t_1, k; t) \;=\; G(k, t_2 - t_1) \;+\; K_1(t_2, t_1, k; t) \;+\; K_2(t_2, t_1, k; t) \;+\; \cdots$$
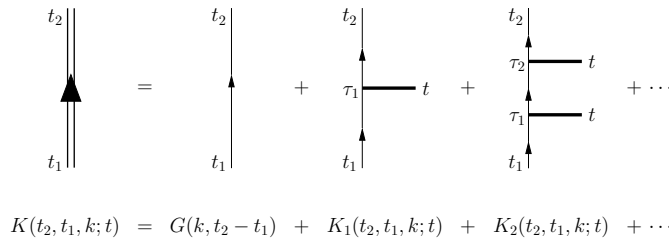
Figure 4. The one-particle Green function with insertions of trees.

calculated previously in [11], hence we have only to calculate one-loop correction to the critical GF.

The generic topological structure of a one-loop diagram is shown in figure 3. To sum up all such diagrams, we need the one-particle Green function with insertions of trees whose end points are located at the same final time t. The graphical representation of such calculations is shown in figure 4.

At zero mass, the one-particle Green function with insertions of p trees is

$$K_p(t_2, t_1, k; t) = (\alpha g)^p \int_{t_1}^{t_2} d\tau_1 \int_{\tau_1}^{t_2} d\tau_2 \cdots \int_{\tau_{p-1}}^{t_2} d\tau_p \, T(t - \tau_1) \cdots T(t - \tau_p) \exp(-Dk^2(t_2 - t_1))$$

$$= \frac{2^p}{p!} \left[\log \left(\frac{1 + r(t - t_2)}{1 + r(t - t_1)} \right) \right]^p \exp(-Dk^2(t_2 - t_1)), \tag{A.1}$$

where $t_2 > t_1$ is assumed and

$$T(t) = (s - 1)/(1 + rt) \tag{A.2}$$

is the MF solution of the critical GF, i.e., $\bar{F}(s, t)$ at $\mu = 0$ and $r = -(s - 1)\alpha g/2$. Hence, the sum over all tree insertions is

$$K(t_2, t_1, k; t) = \sum_{p=0} K_p(t_2, t_1, k; t)$$

$$= \theta(t_2 - t_1) \left(\frac{1 + r(t - t_2)}{1 + r(t - t_1)} \right)^2 \exp(-Dk^2(t_2 - t_1)). \tag{A.3}$$
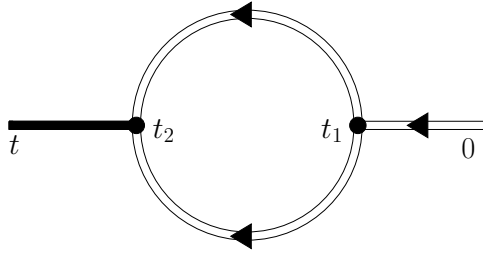
Figure 5. The diagram of one-loop corrections to the GF. The thick line whose terminal point is t is the MF GF and the double line with arrow is the one-particle Green function with insertions of all trees.

With the above Green function the sum over all one-loop diagrams has the form (see also figure 5).

$$L = -\frac{1}{2}\frac{g^2}{(8D\pi)^{d/2}}\int_0^t dt_1 \int_{t_1}^t dt_2\, T(t-t_2)K^2(t_2,t_1,0;t)K(t_1,0,0;t)(t_2-t_1)^{-d/2}, \quad (A.4)$$

where the integral over momentum was done. The integrals in the foregoing expression are standard and we have, using dimensional regularization,

$$L = -\frac{u^2(s-1)}{(1+rt)^2}\left[-\frac{1}{1-\epsilon/2}\left(\frac{2}{\epsilon}\left(\frac{t}{\bar{t}}\right)^{\frac{\epsilon}{2}} + rt\left(\frac{t}{\bar{t}}\right)^{\frac{\epsilon}{2}}\right)\right.$$

$$-3\frac{2}{\epsilon}\frac{rt}{1+\epsilon/2}\left(\frac{t}{\bar{t}}\right)^{\frac{\epsilon}{2}} + 3\frac{1}{1+\epsilon/2}(rt - \log(1+rt))$$

$$\left. -\frac{1}{2+\epsilon/2}\left(2rt - 2\log(1+rt) - \frac{r^2t^2}{1+rt}\right) + O(\epsilon)\right], \quad (A.5)$$

where $\epsilon = 4-d$, $u = \sqrt{K_4}(g/D)\kappa^{-\epsilon/2}/4$ with $K_4 = 1/(8\pi^2)$ is the dimensionless coupling constant and $\bar{t}^{-1} = 8\pi D\kappa^2$ with κ^{-1} an arbitrary length scale. The renormalized GF can be obtained using the standard renormalization constants $Z_\phi = 1+(2/\epsilon)u^2$ and $Z_g = 1+(5/\epsilon)u^2$; see equation (16). Hence, the renormalized GF up to one-loop order is

$$\bar{F}(s,t) = \frac{(s-1)}{1+rt}\left[1 + \frac{u^2}{1+rt}\left(1 + \log(t/\bar{t}) + 2\log(1+rt) + rt\left(3\log(t/\bar{t}) - \frac{4+\frac{9}{2}rt}{1+rt}\right)\right)\right]. \quad (A.6)$$

For the calculation of the logarithmic correction we use the Wilson functions in two-loop order from [11]

$$\beta(u) = 6u^3 - \frac{1}{4}\left(169 + 106\ln\frac{4}{3}\right)u^5, \quad (A.7)$$

$$\gamma_\phi(u) = -2u^2 + 2\left(6 - 9\ln\frac{4}{3}\right)u^4, \quad (A.8)$$

$$\gamma_D(u) = -u^2 + \frac{1}{4}\left(17 - 2\ln\frac{4}{3}\right)u^4, \quad (A.9)$$

with the observation that in [11] the Wilson functions are written for the square of our coupling constant multiplied by a different factor.

The characteristic system to be solved is (see equations (35), (36))

$$\rho\frac{du}{d\rho} = \beta(u), \quad (A.10)$$

$$\rho \frac{d\alpha(\rho)}{d\rho} = (-2 + \gamma_\phi(u)/2)\alpha(\rho), \tag{A.11}$$

$$\rho \frac{dt(\rho)}{d\rho} = (2 + \gamma_D(u))t(\rho). \tag{A.12}$$

Using $d\rho/\rho = dw/\beta(w)$, we have

$$t(w) = tC_t w^{a_0} \exp[-1/(\beta_3 w^2) + b_0 w^2 + O(w^4)], \tag{A.13}$$

$$\alpha(w) = C_\alpha w^{a_1} \exp[1/(\beta_3 w^2) + b_1 w^2 + O(w^4)], \tag{A.14}$$

$$\int_u^w \frac{\gamma_\phi(w')}{\beta(w')} \, dw' = N' w^{a_2} \exp[b_2 w^2 + O(w^4)], \tag{A.15}$$

where C_t, C_α are determined from the initial condition $t(w = u) = t, \alpha(w = u) = \alpha$. We denote by β_i, ϕ_i and D_i the coefficients in the expansion of the Wilson functions $\beta(w)$, $\gamma_\phi(w)$ and $\gamma_D(w)$ that generically can be written as $Q = \sum_i c_{Q,i} u^i$, similar to the notation used in [14]. The constants $a_0, a_1, a_2, b_0, b_1, b_2$ are defined by the following relations:

$$a_0 = \frac{1}{\beta_3^2}(D_2 \beta_3 - 2\beta_5), \tag{A.16}$$

$$a_1 = \frac{1}{2\beta_3^2}(\phi_2 \beta_3 + 4\beta_5), \tag{A.17}$$

$$a_2 = \phi_2/\beta_3, \tag{A.18}$$

$$b_0 = \frac{1}{2\beta_3^2}(D_4 \beta_3 - D_2 \beta_5) + \frac{\beta_5^2}{\beta_3^3}, \tag{A.19}$$

$$b_1 = \frac{1}{4\beta_3^2}(\phi_4 \beta_3 - \phi_2 \beta_5) - \frac{\beta_5^2}{\beta_3^3}, \tag{A.20}$$

$$b_2 = \frac{1}{2\beta_3^2}(\phi_4 \beta_3 - \phi_2 \beta_5). \tag{A.21}$$

Under scaling the generating function transforms as follows:

$$\bar{F}(s, t; w, \alpha) = \exp\left[\int_u^w dw' \, \gamma_\phi(w')/\beta(w')\right] \bar{F}(s, t(w); w, \alpha(w)), \tag{A.22}$$

where we can eliminate w with the condition $t(w) = X_0 \bar{t}$ with X_0 an arbitrary constant of order 1. The GF at large t is

$$\bar{F}(s, t) = \frac{N'(s - 1)w^{a_2} \exp(b_2 w^2 + O(w^4))(1 + (\ln X_0 + 1)w^2)}{S(w, s, t)}$$

$$\times \left[1 + \frac{w^2}{S(w, s, t)}\left(1 + \log X_0 + 2\log(S(w, s, t))\right.\right.$$

$$\left.\left. - (1 - S(w, s, t))\left(3\log X_0 - \frac{9}{2} + \frac{1}{2S(w, s, t)}\right)\right)\right], \tag{A.23}$$

where

$$S(w, s, t) = 1 - \tfrac{1}{2}(s - 1)\bar{D}t w^{a_0 + a_1 + 1} \exp((b_0 + b_1)w^2 + O(w^4)), \tag{A.24}$$

$\bar{D} = C_\alpha C_t D K_4^{-1/2}$ and for large t we have from equation (A.13)

$$w = \tau^{-1/2} \exp\left[\frac{a_0 \beta_3}{4} \frac{\ln \tau}{\tau} + O(\tau^{-2} \ln^2 \tau)\right], \tag{A.25}$$

$$\tau = \beta_3 \ln(t/t_0). \tag{A.26}$$

with $t_0 = (X_0/C_t)\bar{t}$.

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